8 Relations

8.1 Relations and Their Properties

1. for sets A and B we define a relation from A to B to be a subset of $A \times B$

2. for example the graph of a function is a relation from the domain (A) to its range (B). Note that every function a relation because it relates an element of A to an element of B. However not every relation a function because a relation may relate an element of A to more than just one element of B (or also because the empty relation is not a function).

3. a relation on a set A is a relation from A to A (i.e. subset of $A \times A$, where A could be finite or infinite)

4. for a finite set A (with $|A| = n$), there are $2^{|A \times A|} = 2^{n^2}$ possible relations on A, namely the elements of the powerset of $A \times A$

5. a relation R defined on A is reflexive if $(a, a) \in R, \forall a \in A$

6. a relation R defined on A is symmetric if for elements $a, b \in A$ we have that if $(a, b) \in R$ then $(b, a) \in R$

7. a relation R defined on A is antisymmetric if for elements $a, b \in A$ we have that if $(a, b) \in R$ and $(b, a) \in R$ then $a = b$

8. a relation is transitive R defined on A if for elements $a, b, c \in A$ we have that if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$

9. a relation is an equivalence relation if it is reflexive, symmetric and transitive

10. operations on relation are operations on sets (like union, intersection, difference, symmetric difference ($R \oplus S = (R - S) \cup (S - R)$))

11. composition of relations: for elements $a, b, c \in A$, and two relations R and S, if $(a, b) \in R$ and $(b, c) \in S$ then $(a, c) \in S \circ R$

12. composing a relation to itself: $R^n = R^{n-1} \circ R$ with the initial condition $R^1 = R$

13. a relation is transitive iff $R^n \subseteq R$

8.3 Representing Relations

representing relations using matrices

1. a zero-one matrix $M_R = [m_{ij}]$ can be used to represent a relation $R = \{(a_i, b_j)$ for some $i, j\}$ if we let $m_{ij} = 1$ iff $(a_i, b_j) \in R$.
2. if R is defined on a set, then M_R is a square matrix.
3. R is reflexive iff M_R has only 1s on its diagonal
4. R is irreflexive iff M_R has only 0s on its diagonal
5. R is symmetric iff $M_R = (M_R)^t$ (i.e. M_R is a symmetric matrix)
6. R is antisymmetric iff M_R has either $m_{i,j} = 0$ or $m_{j,i} = 0$ (or both) for all $i \neq j$
7. $M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2}$ and $M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2}$ and $M_{R_1 \circ R_2} = M_{R_2} \odot M_{R_1}$ and $M_{R^n} = M_R^{[n]}$

representing relations using digraphs

1. a digraph consists of a set V of vertices (nodes) and a set E of ordered pairs (arcs): if $a, b \in V$ and $(a, b) \in E$ then the arc (a, b) (or just (ab)) belongs to the digraph, where a is the initial vertex and b is the terminal vertex of the arc
2. the arc (a, a) is a loop
3. a relation can be modeled using a digraph where each arc of the digraph represents an element of the relation
4. a relation is reflexive iff there is a loop at every vertex
5. a relation is irreflexive iff there are no loops
6. a relation is symmetric iff every time the arc (a, b) is present, the arc (b, a) is present (if both arc (b, a) and (a, b) are presents, the two arcs may be replaced by one undirected edge)
7. a relation is antisymmetric iff for every pair of vertices a and b, at most one of the arcs (a, b) and (b, a) is present
8. a relation is transitive iff whenever the arcs (a, b) and (b, c) are present, then the arc (a, c) is present (make sure you use the loops in the case you have the arcs (a, b) and (b, a) since you'll need to have the loops (a, a) and (b, b))

8.5 Equivalence Relations

1. an equivalence relation R on A is a relation that is reflexive, symmetric and transitive
2. for an equivalence relation R, the equivalence class $[a]_R$ (or simply $[a]$) of an element $a \in A$ is the set containing a together with all elements related to a:

$$[a] = \{s : (a, s) \in R\} = \{s : (s, a) \in R\}$$

3. any element of the class can be a representative of the class, and so any element of the class can give the name of the class (the name of an equivalence class is not unique)
4. in particular, the equivalence classes for the equivalence relation "congruence modulo m" are called congruence classes modulo m
5. a partition of a set A is a collection of subsets A_i $(1 \leq i \leq t)$ of A such that
   - any two subsets are disjoint (i.e. $A_i \cap A_j = \emptyset, \forall i \neq j, 1 \leq i \neq j \leq t$)
   - no subset is empty (i.e. $A_i \neq \emptyset, \forall i, 1 \leq i \leq t$)
   - the union of the subsets is A itself $\left(\text{i.e. } A = \bigcup_{i=1}^{t} A_i\right)$
6. the equivalence classes of an equivalence relation R on A form a partition of the set A
7. similarly, a partition of A induces an equivalence relation R whose equivalence classes are the partition sets (and thus the relation can be obtained)